INTRODUCTION TO RANDOM SIGNALS

POWER SPECTRAL ANALYSIS

P.R. WHITE

DEFINITION OF THE PSD

TYPES OF POWER SPECTRA

THE WIENER-KHINTCHINE THEOREM

SPECTRAL ESTIMATION ALGORITHMS

BIAS IN SPECTRAL ESTIMATION

VARIANCE OF SPECTRAL ESTIMATORS

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GENERAL REMARKS

A correlation function represents the character of a random process in the time domain.

It is natural to seek a similar representation that represents a random signal as a function of frequency.

Such a representation is the Power Spectral Density (PSD), also referred to as the Power Spectrum, Auto-Spectral Density or simply the Spectrum.

It is a real, positive valued function.

The PSD is probably the most widely used tool for analysing random processes.

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A BASIC PROBLEM

If we are considering a deterministic signal $x(t)$ then one describes the signal in the frequency domain through the Fourier transform $X(f)$.

For $X(f)$ to exist then the signal must have finite energy, *i.e.*

$$\int_{-\infty}^{\infty} x(t)^2 \, dt < \infty$$

For the above to be true the signal $x(t)$ must tend to zero as $t \rightarrow \pm\infty$ and so:

$$|X(f)|^2 = \underset{T \rightarrow \infty}{Lim} \left| \int_{-T}^{T} x(t) e^{-2\pi i f t} dt \right|^2$$

Firstly $|X(f)|^2$ increases in proportion to T, this is normalised out by considering:

$$|X(f)|^2 / T$$

Even so, for a stationary random signal this convergence does not occur and the straightforward use of the Fourier transform is possible.

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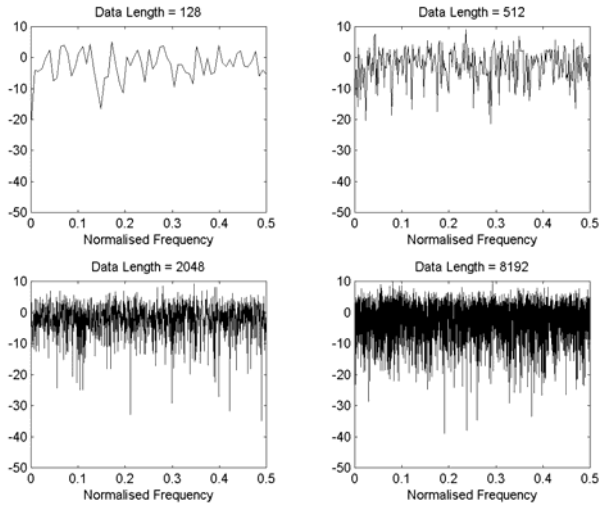
EXAMPLE

Consider $|X(f)|^2 / T$ computed for a white noise process using different data lengths.



As T increases $|X(f)|^2 / T$ fails to converge to an answer.

Increasing in data length serves to increase the number of samples in the output Fourier transform but does not make the result less variable.

In the statistical jargon this is said to be an inconsistent estimator.

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DEFINITION OF THE PSD

One definition of the PSD, $S(f)$, is:

$$\lim_{T \to \infty} \frac{E\left[\left|X_T(f)\right|^2\right]}{T}$$

where $X_T(f)$ is the Fourier transform of $x_T(t)$ which is a T second segment of $x(t)$:

$$x_T(t) = x(t) \qquad -T/2 \le t \le T/2$$
$$= 0 \qquad \text{Elsewhere}$$

Note that this definition of the PSD contains two limiting operations:

- The limit as $T \to \infty$ which serves to increase the resolution of the Fourier transform.
- The ensemble average, $E[]$, that smoothes fluctuations in the spectrum.

The roles of these two operations will be a recurrent theme throughout this presentation.

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DIMENSIONAL ANALYSIS/UNITS OF MEASUREMENT

Assume that $x(t)$ has the units of Volts, as is the case in the majority of applications.

Then the Fourier transform, $X(f)$, has the dimensions of Volts×seconds.

So that the PSD has the dimensions of Volts^2×seconds and is usually measured in the units of v^2/Hz.

This emphasises the fact that the PSD is a density and is not a direct measure of power.

To meaningfully use the PSD to obtain measures of power one needs to employ some integration in frequency.

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SYMMETRY IN THE PSD

The PSD is based on the Fourier transform.

If the signal $x(t)$ is real then is Fourier transform $X(f)$ has the following symmetry:

$$X(-f) = X(f)^*$$

So that:

$$|X(-f)|^2 = |X(f)|^2$$

This leads to the same symmetry property appearing in the PSD, so that:

$$S(-f) = S(f)$$

The function $S(f)$ is sometimes referred to as the two-sided PSD.

An alternative is to define a one-sided spectrum, $G(f)$, that is related to the two-sided spectrum through:

$$\begin{aligned} G(f) &= 2S(f) & f>0 \\ &= S(f) & f=0 \\ &= 0 & f<0 \end{aligned}$$

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PARSEVAL'S THEOREM

It is comparatively straightforward to show that:

$$\int_{-T/2}^{T/2} x_T(t)^2\,dt = \int_{-\infty}^{\infty} |X_T(f)|^2\,df$$

Normalising and taking expectations leads to:

$$\frac{1}{T} E\left[\int_{-T/2}^{T/2} x_T(t)^2\,dt\right] = \frac{1}{T} E\left[\int_{-\infty}^{\infty} |X(f)|^2\,df\right]$$

$$\frac{1}{T}\int_{-T/2}^{T/2} E\left[x_T(t)^2\right]dt = E\left[x_T(t)^2\right] = \frac{1}{T}\int_{-\infty}^{\infty} E\left[|X(f)|^2\right]df$$

$$P = \int_{-\infty}^{\infty} \underset{T\to\infty}{Lim}\frac{1}{T} E\left[|X(f)|^2\right]df = \int_{-\infty}^{\infty} S(f)\,df$$

Hence integrating the spectrum over all frequencies yields the total signal power, *P*.

This observation allows one to regard the spectrum as a decomposition of a signal's power over frequency.

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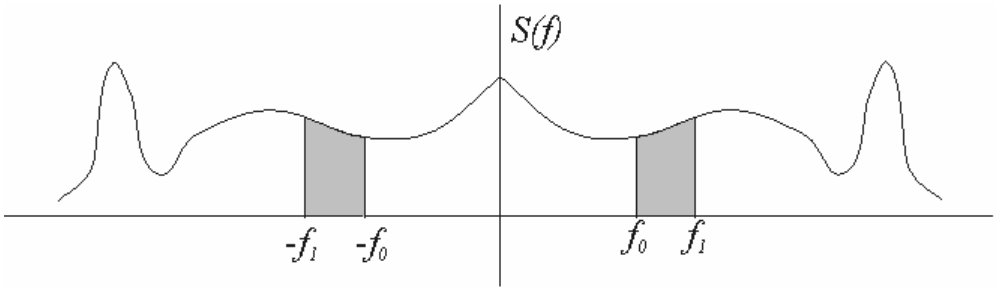
USING THE PSD

The simplest use of the PSD is to estimate the power of a signal within a frequency band.

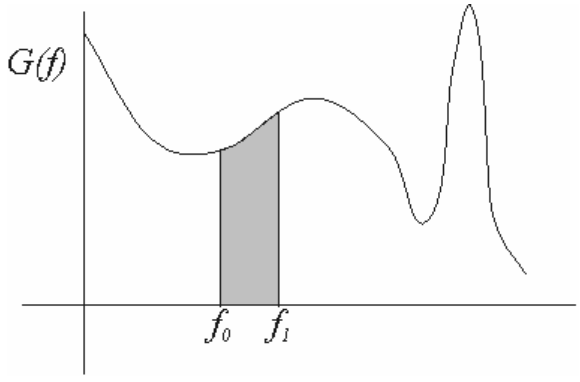
Specifically to compute the power between f_0 and f_1 using the two-sided spectrum one calculates:

$$\int_{f_0}^{f_1} S(f)df + \int_{-f_1}^{-f_0} S(f)df = 2\int_{f_0}^{f_1} S(f)df$$



Alternatively using the one-sided spectrum the computation of the power in the band f_0 to f_1 is given by:

$$\int_{f_0}^{f_1} G(f)df$$



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TYPES OF POWER SPECTRA

Continuous Power Spectra

Most *broad-band* random processes posses a power spectrum that is a continuous function of frequency.

For example if white noise is filtered with a fixed linear filter then the output process's spectrum will be continuous.

Discrete Spectra

A random tonal signal has a spectrum that contains Dirac delta functions and so is not continuous.

A Dirac delta being a "function" of zero width and infinite height.

Mixed Processes

In many practical applications one encounters processes that contain both tonal and broad-band components.

In which case spectrum is continuous almost everywhere, with the exception of the discrete spectral lines.

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RELATIONSHIP TO CORRELATION FUNCTIONS

Recall:

$$r(\tau) = r_{xx}(\tau) = E\left[x(t)x(t-\tau)\right]$$

For an ergodic signal one also has:

$$r_{xx}(\tau) = \underset{T \to \infty}{Lim} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)dt$$

From this definition one can show the following relationship between the PSD and correlation function:

$$S(f) = \int_{-\infty}^{\infty} r_{xx}(\tau)e^{-2\pi i f \tau}d\tau$$

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} S(f)e^{2\pi i f \tau}df$$

This is the *Wiener-Khintchine theorem* and states that the PSD and correlation functions represent a Fourier transform pair.

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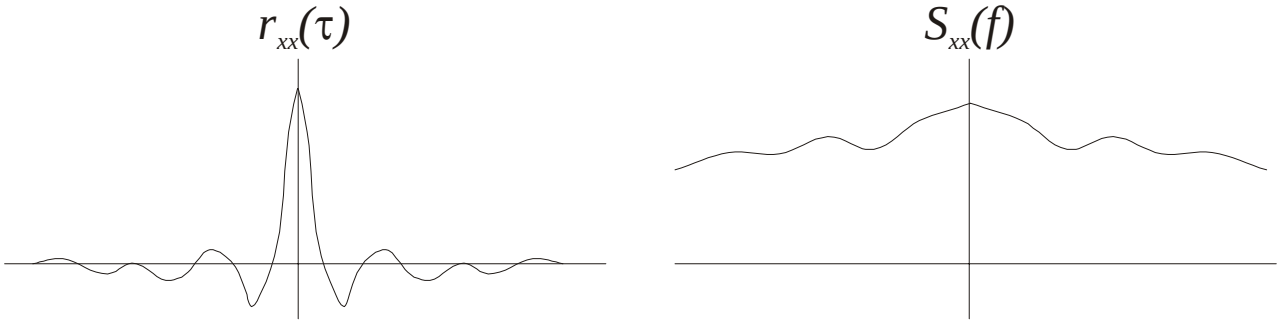
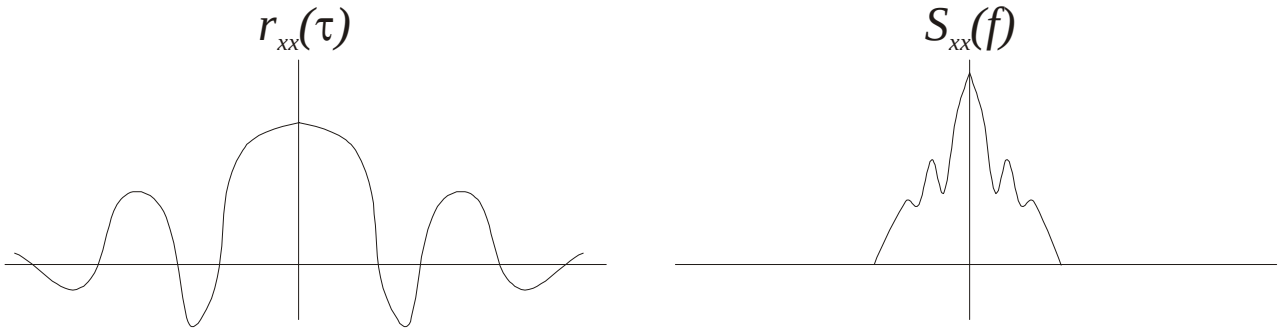
IMPLICATIONS OF WIENER-KHINTCHINE THEOREM

Highly erratic signals have correlation functions that are only large close to the origin and decay rapidly, but they contain much high frequency energy, so their PSD is spread.



Slowly varying signals have correlation functions that decay slowly away from the origin, but they are dominated by low frequencies.



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EXAMPLE: MARKOV PROCESS

Recall that for the Markov process:

$$r(m) = (2\alpha - 1)^m, \qquad m \geq 0$$

Using the Fourier transform of a sequence the spectrum is defined as:

$$S(f) = \sum_{m=-\infty}^{\infty} r(m) e^{-2\pi i f m \Delta}$$

where Δ is the sampling interval.

Defining $\rho=(2\alpha-1)$ the spectrum is:

$$S(f) = 1 + \sum_{m=1}^{\infty} \rho^m e^{-2\pi i f m \Delta} + \sum_{m=-1}^{-\infty} \rho^{-m} e^{-2\pi i f m \Delta}$$

$$S(f) = 1 + \frac{\rho e^{-2\pi i f m \Delta}}{1 - \rho e^{-2\pi i f m \Delta}} + \frac{\rho e^{2\pi i f m \Delta}}{1 - \rho e^{2\pi i f m \Delta}}$$

$$S(f) = \frac{1 - \rho^2}{1 + 2\rho \cos(2\pi f \Delta) - \rho^2}$$

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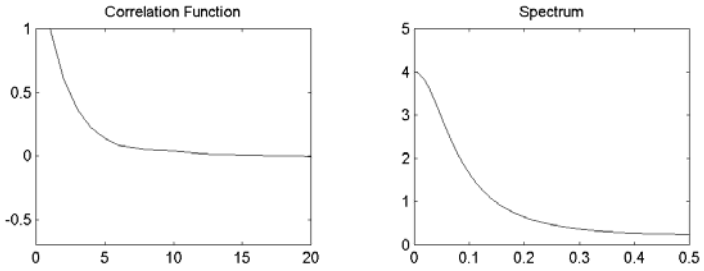
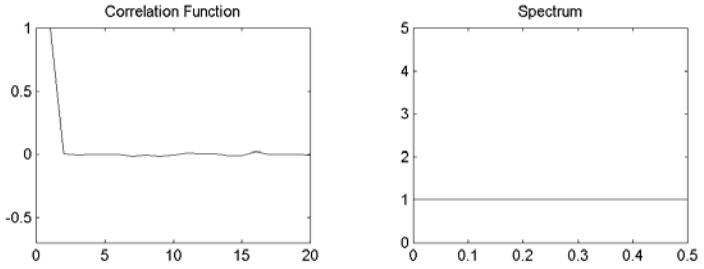
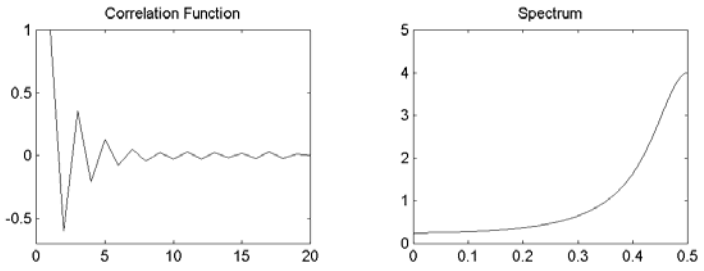
1) For large values of α (close to 1) the value of ρ is close to 1 and the correlation function is slowly decaying and the spectrum is narrow and is centred on zero.



2) For α=0.5, then ρ=0, so the correlation function is an impulse and the spectrum is flat, *i.e.* the signal is a white noise process.



3) For small α the value of ρ is close to –1, the correlation function is slowly decaying, but rapidly oscillating, and the spectrum is narrow and is centred on $f_s/2$.



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SPECTRAL ESTIMATION SCHEMES

In practice one needs algorithms for constructing an estimate of the spectrum $S(f)$ from a finite record of data $x(t)$.

There are two broad classes of spectral estimation algorithm:

Non-Parametric

Based on no specific model of the underlying process, so are widely (universally) applicable.

Parametric

Exploits a specific model of the stochastic process, *e.g.* the process is modelled as filtered white noise.

Assuming the underlying model is appropriate then the parametric methods tend to outperform non-parametric methods.

We shall only consider non-parametric approaches.

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TYPES OF NON-PARAMETRIC SPECTRAL ESTIMATORS

There are 3 types of non-parametric methods we shall consider:

1) *Filter bank methods*

Basis of analogue spectral analysers.

2) *Correlation based methods*

Method favoured by many statisticians, based on the Wiener-Khintchine theorem.

3) *The direct/segment averaging approach*

This is the technique that is almost universally used in modern digital spectral analysers.

We shall discuss each of these in the following.

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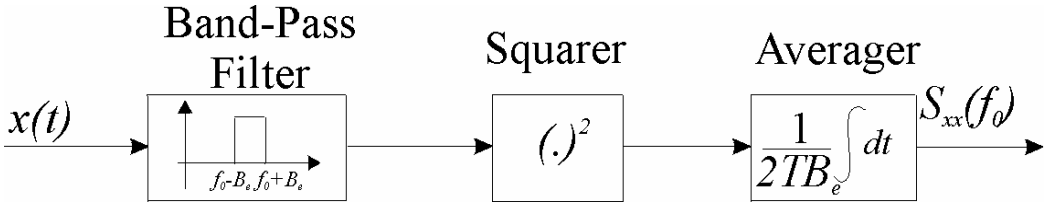
FILTER BANK METHOD

The basis of this method is that the spectrum $S(f_0)$ reflects the power in a frequency band close to f_0.

It consists of the following steps:

1) Filter the signal to restrict it to the band $f_0 \pm B_e/2$ (B_e is the filter's bandwidth).
2) Square the signal.
3) Integrate the squared signal.
4) Normalise as appropriate.

The result is an estimate the power of the signal in the frequency band $f_0 \pm B_e/2$ which, assuming B_e is sufficiently small, is an estimate of $S(f_0)$.



Using a parallel set of the circuits, tuned to different centre frequencies, allows one to estimate the complete spectrum.

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PARAMETER CHOICE

When designing such a filter bank scheme one needs to select several basic parameters:

The Filter Bandwidth B_e

If one chooses B_e to be small then the complexity of the system increases.

The Integration Time T

Increasing T results in a smoother estimate.

Note that as the bandwidth, B_e, reduces it becomes necessary to increase T.

Filter Characteristic

One seeks to use filters with sharp transitions. Generally the phase response is unimportant since the power estimate is "phase blind".

The filter bank scheme is ideally suited to analogue implementation.

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CORRELATION BASED METHODS

This method is based on the Wiener-Khintchine theorem:

$$S(f) = \int_{-\infty}^{\infty} r(\tau) e^{-2\pi i f \tau} d\tau$$

Methods for estimating the correlation function, $r(\tau)$, have already been discussed.

One approach for estimating the spectrum is to estimate the correlation function, $r(\tau)$, and then Fourier transform it to obtain an estimate of the spectrum.

Two correlation estimators have been considered:

1) The unbiased estimator, $\tilde{r}(\tau)$:

$$\tilde{r}(\tau) = \frac{1}{T-\tau} \int_{\tau}^{T} x(t)\, x(t-\tau)\, dt, \quad \tau > 0$$

2) The biased estimator, $\hat{r}(\tau)$:

$$\hat{r}(\tau) = \frac{1}{T} \int_{\tau}^{T} x(t)\, x(t-\tau)\, dt, \quad \tau > 0$$

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A General Correlation Based Method

Recall the connection between the biased and unbiased estimators:

$$\hat{r}(\tau) = \left(1 - \frac{|\tau|}{T}\right)\tilde{r}(\tau)$$

The biased estimate is obtained by multiplying the unbiased estimate by a triangular shaped windowing function.

The triangular shape coming from the convolution of the two rectangular windows used to form $x_T(t)$.

As previously discussed the biased estimator is often preferred to the unbiased estimator.

A general correlation estimator, $\bar{r}(\tau)$, is obtained by windowing the biased (or unbiased) correlation estimator:

$$\bar{r}(\tau) = w(\tau)\hat{r}(\tau)$$

where $w(\tau)$ is a function that is zero for $|\tau| > T/2$.

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THE EFFECT OF WINDOWING THE CORRELATION FUNCTION

The general correlation based spectral estimator, $\overline{S}(f)$, is:

$$\overline{S}(f) = \int_{-T/2}^{T/2} \overline{r}(\tau) e^{-2\pi i f \tau} d\tau = \int_{-T/2}^{T/2} w(\tau)\hat{r}(\tau) e^{-2\pi i f \tau} d\tau$$

This can expressed in the frequency domain as:

$$\overline{S}(f) = W(f) * \hat{S}(f)$$

where

$$\hat{S}(f) = \frac{|X_T(f)|^2}{T}$$

This function $\hat{S}(f)$ is referred to as the 'raw spectral estimate'.

We have already discussed the properties of this estimator, specifically that it fails to converge for increasingly large values of T.

The effect of convolving $\hat{S}(f)$ by $W(f)$ is to perform local averaging of the spectrum.

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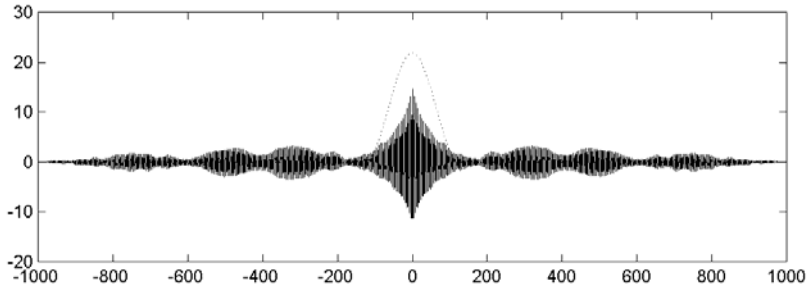
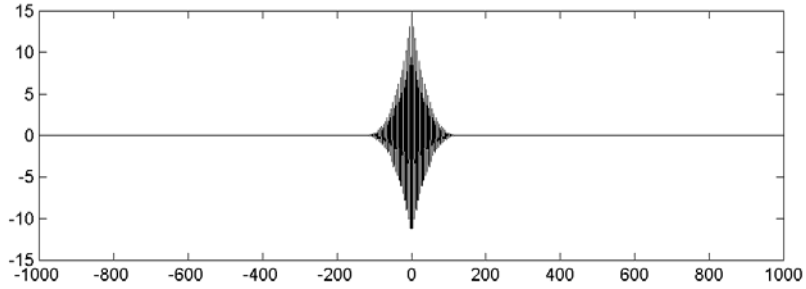
PICTORIAL REPRESENTATION OF CORRELATION BASED METHODS

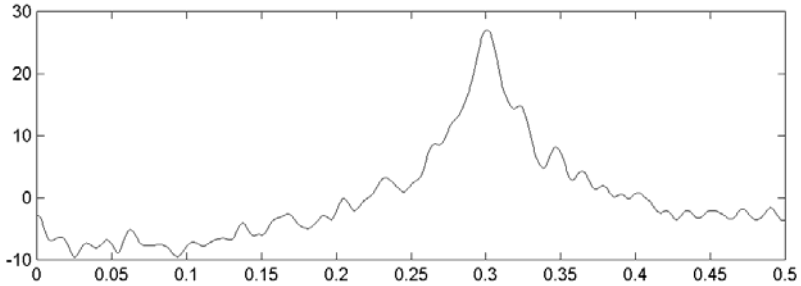
1) Compute the raw correlation function:



2) Multiply by the windowing function:



3) Fourier transform:



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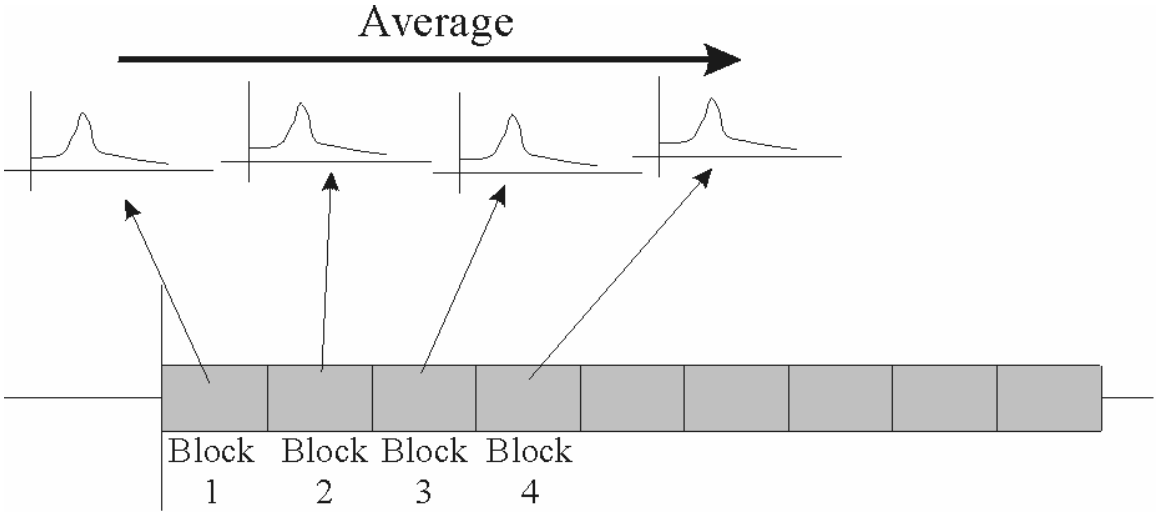
SEGMENT AVERAGING

The third approach to spectral estimation is segment averaging.

This is based on breaking the signal into small blocks of *M* samples, these blocks maybe overlapped.

Each block of data is Fourier transformed, normally after windowing.

The squared magnitudes of these Fourier transforms are then averaged.



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FORMULATION OF SEGMENT AVERAGING

Let each block of data, $x_p(t)$, be defined as (we shall assume that there is no overlapping only for notational simplicity):

$$x_p(t) = x(t) \qquad (p-1)K < t \le pK$$
$$= 0 \qquad \text{Elsewhere}$$

where K is the duration of a block.

This data is windowed and Fourier transformed:

$$X_p(f) = \mathsf{F}\{w(t)x_p(t)\}$$

The spectral estimator is then formed as:

$$\hat{S}(f) = \frac{1}{MK}\sum_{p=1}^{M} \left|X_p(f)\right|^2$$

Note that in this method a windowing function is applied to the time sequence, $x(t)$, whereas in the correlation based method the window is applied to the correlation function (which is a quadratic function of the signal).

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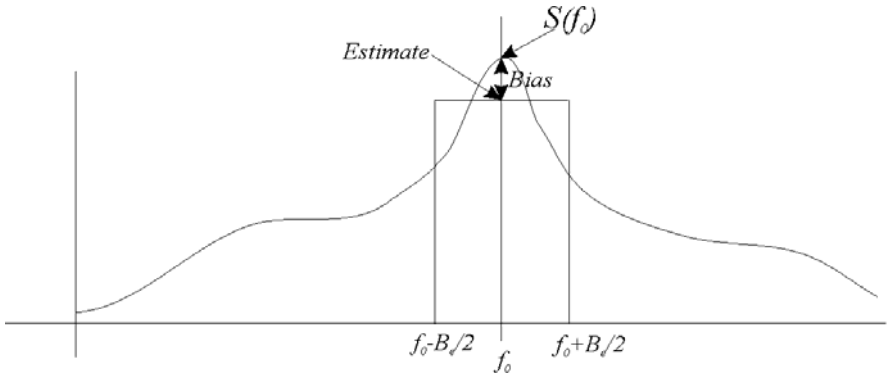
BIAS IN SPECTRAL ESTIMATORS

Recall the bias of any estimator, θ, is: $b = \theta - E[\theta]$

For simplicity we shall estimate the bias for the filter bank approach.



The power of the output of the filter will be the integral of the spectrum over $f_0 \pm B_e/2$, this is represented by the height of the rectangle in the figure.

Clearly if the spectrum is approximately flat over the filter's bandwidth then the bias will be small.

Further if the spectrum is linear over the filter's bandwidth the bias will also be zero.

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EXPRESSION FOR THE BIAS

Expand the true spectrum $S(f)$ as a Taylor series about the point f_0 and retain only the first 3 terms:

$$S(f) \approx S(f_0) + (f - f_0)S'(f_0) + \frac{(f - f_0)^2}{2} S''(f_0)$$

The area under the curve between f_0-$B_e/2$ and f_0+$B_e/2$ is given by:

$$\int_{f_0 - B_e/2}^{f_0 + B_e/2} S(f)\,df$$

Substituting the Taylor series expansion (and simplifying) gives:

$$B_e \left\{ S(f_0) + \frac{B_e^{\,2}}{24} S''(f_0) \right\}$$

From which it is straightforward to show that the bias in the estimate is given by:

$$\frac{B_e^{\,2}}{24} S''(f_0)$$

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COMMENTS ON BIAS EXPRESSION

1) Although it is derived from the filter bank method the principles are equally applicable to all spectral estimation schemes.

2) The derivation assumes a “brick wall” filter, this is not possible in practice. Using the equivalent bandwidth, B_e, helps in this matter, since B_e for a filter is defined as the bandwidth of an equivalent brick wall filter.

3) The Taylor series approximation limits the accuracy of this expression.

4) The bias is proportional to the bandwidth, *i.e.* the narrower the filter’s bandwidth the less the bias.

5) The bias is proportional to the second derivative (curvature) of the spectrum:

For a peak the second derivative is negative and so the spectral estimate under estimates the spectrum.

For a trough the second derivative is positive and so the spectral estimate over estimates the spectrum.

The sharper the peak/trough the greater the bias.

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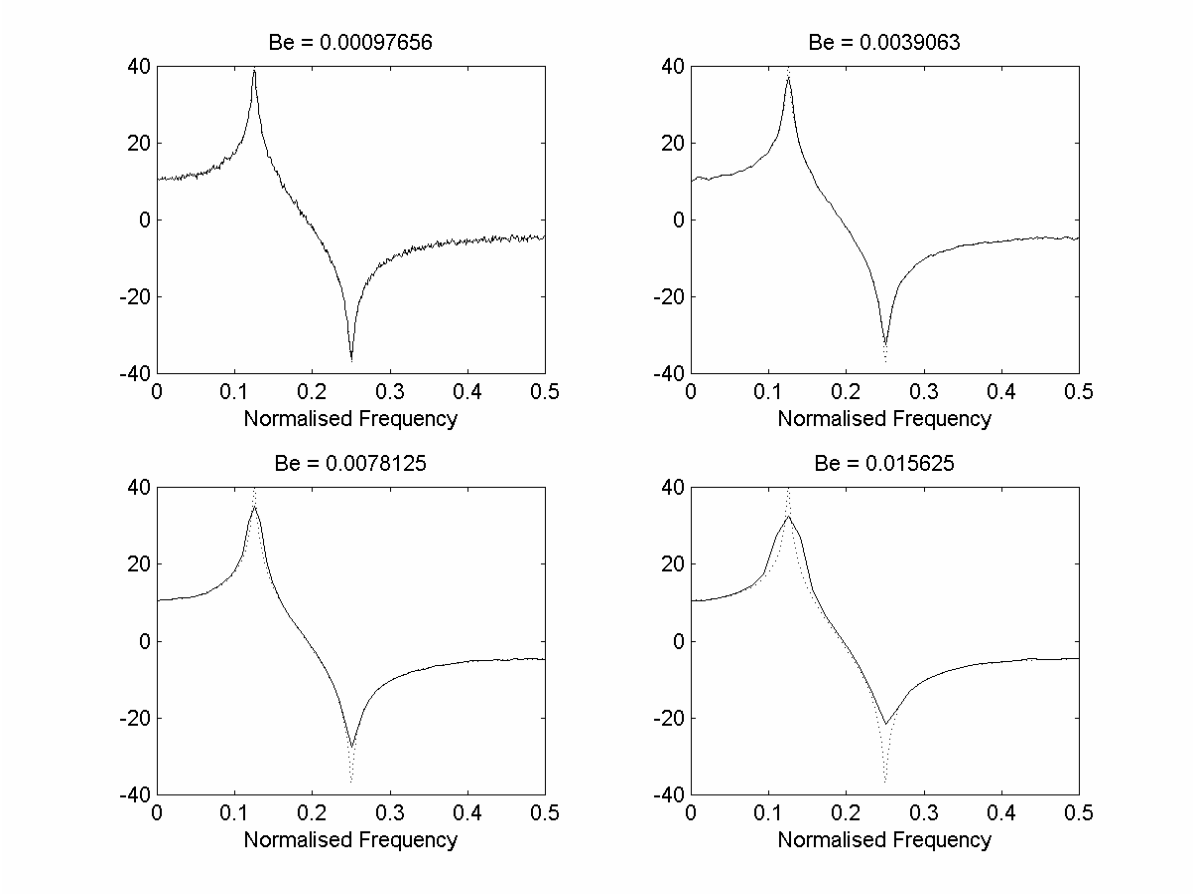
EXAMPLE

The dotted line shows the theoretical spectrum, the solid line shows estimates at different resolutions.



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VARIANCE IN SPECTRAL ESTIMATES

The raw spectral estimate (obtained by Fourier transforming the full data record) has a large variance that does not reduce as T increases.

To reduce this variance some form of smoothing has to be employed:

- In the filter bank method smoothing is achieved through the use of the integrator.
- In the correlation method smoothing is achieved by the convolution of the raw spectrum with $W(f)$.
- In segment averaging smoothing is achieved by the averaging of the segments' raw spectra.

Here we shall consider results in the context of the segment averaging approach.

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VARIANCE OF THE RAW SPECTRUM (I)

Firstly let us consider the raw spectrum:

$$\frac{\left|X_T(f)\right|^2}{T}$$

The Fourier operator can be expressed as follows:

$$X_T(f) = \int_{-T/2}^{T/2} x_T(t) e^{-2\pi i f t} dt$$

$$\mathrm{Re}\left\{X_T(f)\right\} = R(f) = \int_{-T/2}^{T/2} x_T(t) \cos(2\pi f t)\, dt$$

$$\mathrm{Im}\left\{X_T(f)\right\} = I(f) = \int_{-T/2}^{T/2} x_T(t) \sin(2\pi f t)\, dt$$

If $x_T(t)$ is Gaussian then $R(f)$ and $I(f)$ are also Gaussian, but further that they are also statistically independent and we have

$$\left|X_T(f)\right|^2 = R(f)^2 + I(f)^2$$

Further one can show that:

$$E\left[R(f)\right] = E\left[I(f)\right] = 0$$

and

$$E\left[R(f)^2\right] = E\left[I(f)^2\right] = TS(f)/2$$

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VARIANCE OF THE RAW SPECTRUM (II)

From the previous discussions:

$$R(f)\sqrt{\frac{2}{TS(f)}} \quad \text{and} \quad I(f)\sqrt{\frac{2}{TS(f)}}$$

are both Gaussian random variables with mean zero and unit variance (standardised Gaussian r.v.s). So that:

$$\frac{2}{TS(f)}R(f)^2+\frac{2}{TS(f)}I(f)^2=\frac{2}{TS(f)}\left\{R(f)^2+I(f)^2\right\}=\frac{2\left|X_T(f)\right|^2}{TS(f)}$$

is a chi-squared random variable with 2 degrees of freedom.

Recalling that the variance of a chi-squared r.v. is equal to $2k$, where k is the degrees of freedom, so

$$var\left\{\frac{2\left|X_T(f)\right|^2}{TS(f)}\right\}=4 \quad \Rightarrow \quad Var\left\{\frac{\left|X_T(f)\right|^2}{T}\right\}=S(f)^2$$

This emphasises that the variance of the raw spectrum does not reduce with T.

Further note that the variance of the spectral estimate is proportional to the true spectrum.

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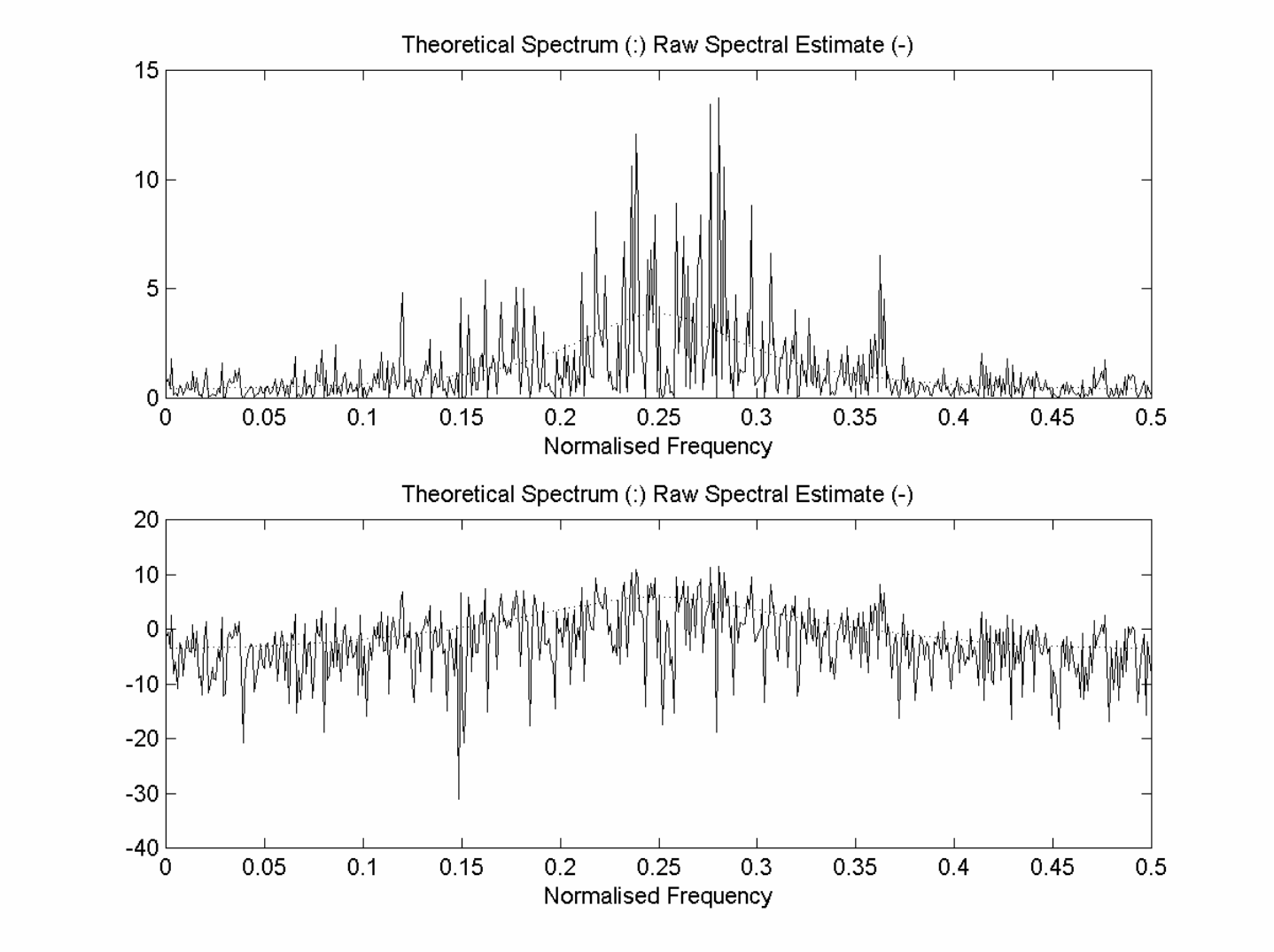
EXAMPLE



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VARIANCE OF SEGMENT AVERAGED SPECTRAL ESTIMATE

We shall consider a simplified segment averaging scheme in which:

1) The data is not windowed

2) No overlapping is used

The principles derived from this scheme extend to more general segment averaging schemes.

The assumption that no overlapping is used allows one to suggest that the data in two blocks will be nearly independent.

If M independent quantities are averaged then their variance is reduced by $1/M$, so the variance of the segment averaged spectral estimate is:

$$\frac{S(f)^2}{M}$$

So to reduce the variance we require M to be large, *i.e.* a large number of segments should be employed.

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VARIANCE OF CORRELATION BASED SPECTRAL ESTIMATE

Correlation based schemes compute:

$$\hat{S}(f) = \int_{-\infty}^{\infty} w(\tau)\hat{r}(\tau)e^{-2\pi i f \tau} d\tau = W(f) * \frac{\left|X_T(f)\right|^2}{T}$$

i.e. the spectral estimate is obtained by convolving the raw spectral estimate with the Fourier transform of the windowing function.

Recall that the raw spectrum is distributed as a chi-squared r.v. with 2 degrees of freedom.

One can show that the results of the correlation based method is also approximately chi-squared distributed with ν degrees of freedom.

The value of ν depends on the window and is given by:

$$\nu \approx \frac{2}{\frac{1}{T}\int_{-\infty}^{\infty} w(t)^2 \, dt}$$

and the variance is:

$$\frac{2S(f)^2}{\nu}$$

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CONFIDENCE INTERVALS

The principle of a confidence interval is similar to the idea of error bars.

One seeks to put limits on the likely errors associated with an estimate.

Consider a general estimator θ, which has a probability distribution $p(\theta)$.

The definition of a confidence interval is:

The range $\left[\hat{\theta}-\Delta\theta_1,\hat{\theta}+\Delta\theta_2\right]$ *that will enclose the true value of* θ *on 100(1-*α*)% of trials.*

Clearly the values $\Delta\theta_1$ and $\Delta\theta_2$ depend on $p(\theta)$ and the value of α.

The values $\Delta\theta_1$ and $\Delta\theta_2$ are computed as the values such that:

$$\int_{-\infty}^{\theta-\Delta\theta_1} p(\theta)d\theta = \int_{\theta+\Delta\theta_2}^{\infty} p(\theta)d\theta = \alpha/2$$

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CONFIDENCE INTERVAL FOR CORRELATION BASED SPECTRAL ESTIMATORS

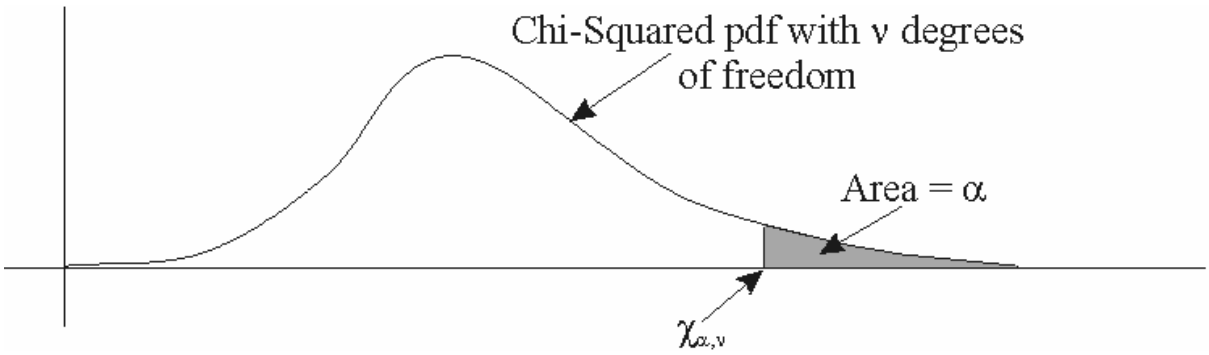
Assuming one is constructing a correlation based estimator the confidence interval is given by:

$$\frac{\nu\hat{S}(f)}{\chi_{\nu,1-\alpha/2}} \leq S(f) \leq \frac{\nu\hat{S}(f)}{\chi_{\nu,\alpha/2}}$$

where $\chi_{\nu,a}$ is given by:

$$\int_{\chi_{\nu,\alpha}}^{\infty} p_{\chi,\nu}(x)\,dx = \alpha$$

$p_{\chi,\nu}(x)$ is the pdf of a chi-squared r.v. with ν degrees of freedom.



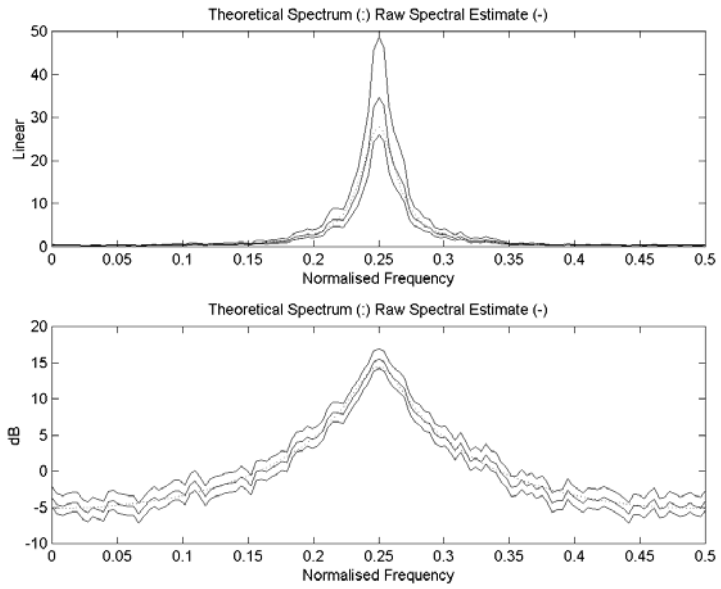
The values of $\chi_{\nu,a}$ are widely tabulated.

CONFIDENCE INTERVAL FOR SEGMENT AVERAGED SPECTRAL ESTIMATORS

A segment averaged spectral estimator is also approximately chi-squared distributed, with $2M$ degrees of freedom (M is the number of segments).

The corresponding confidence interval is:

$$\frac{M\hat{S}(f)}{\chi_{M,1-\alpha/2}} \leq S(f) \leq \frac{M\hat{S}(f)}{\chi_{M,\alpha/2}}$$



The three solid lines in these graphs represent upper and lower bounds for the confidence intervals and the spectral estimate.

Note that on a linear scale the confidence intervals are asymmetrical spaced about the spectral estimate, whilst on the dB scale the spacing is symmetric.

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TRADING BIAS AND VARIANCE

All three spectral estimation techniques discussed allow one to reduce variance, but at the expense of increasing the bias.

1) *Filter Bank Method*

To reduce variance one can either increase the integration time or increase the bandwidth of the filters.

Increasing the integration time implies a longer data record.

Increasing the filter bandwidth increases the bias.

2) *Correlation Based Method*

We shall consider this in the frequency domain, so the estimate is formed as the convolution of the raw spectrum with the Fourier transform of the windowing function $W(f)$.

Choosing $W(f)$ to be broad (so $w(\tau)$ is short) leads to greater reductions in variance.

However $W(f)$ smoothes the spectrum so that if it is broad then a greater bias is introduced.

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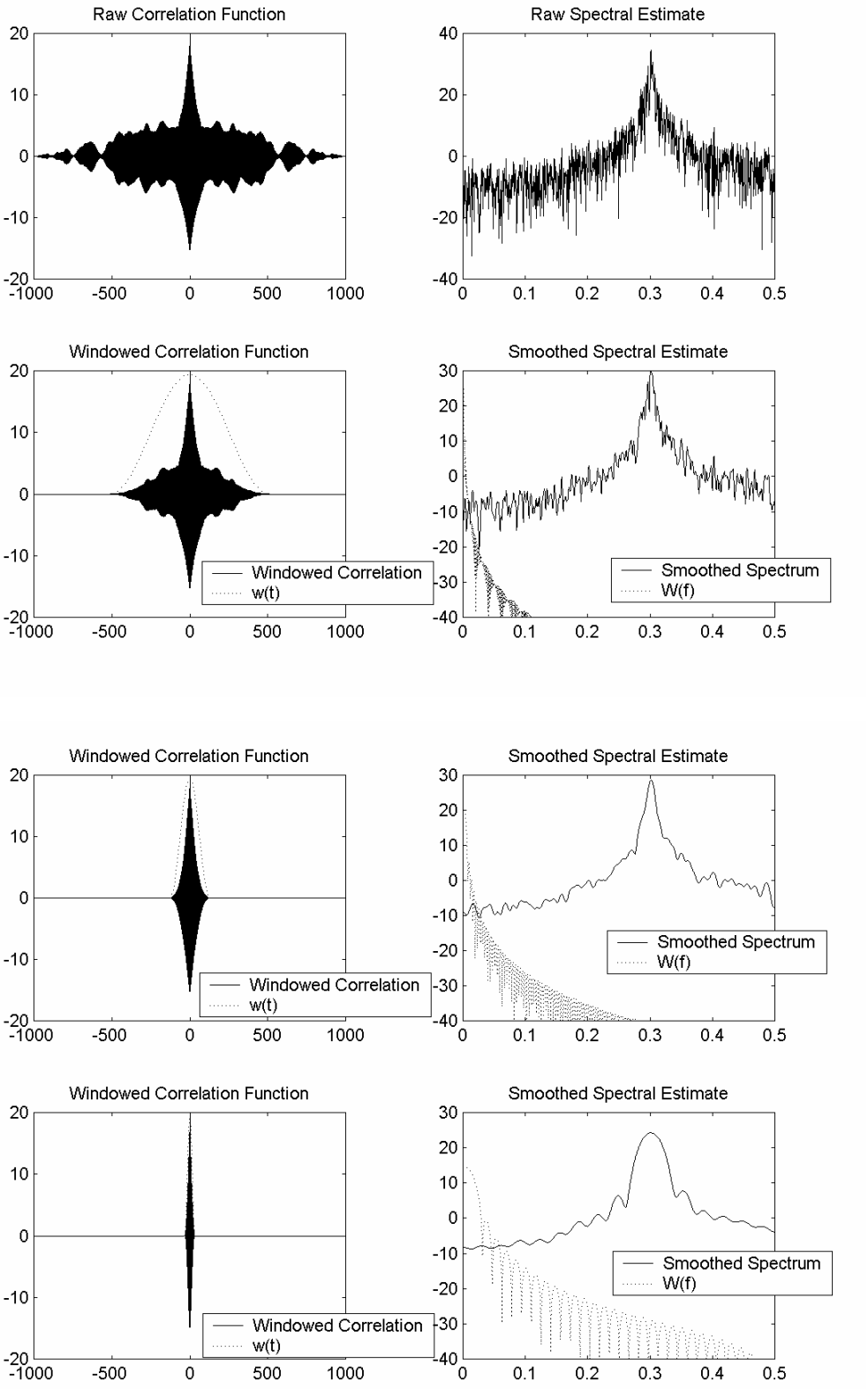
EXAMPLE OF BIAS/VARIANCE TRADE OFF FOR CORRELATION BASED METHODS



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TRADING BIAS AND VARIANCE (II)

3) *Segment Averaging*

Using segment averaging assuming one has a total T seconds of data.

If one chooses a segment length of K then there will be approximately $T/K=M$ independent blocks of data.

If M is large then K will be small. The bias is proportional to the data length of a block, *i.e.* K, but the variance will be small since a lot of averaging will occur.

If M is small then K will be large. Leading to less bias, but also increasing the variability since there will be fewer independent blocks to average over.

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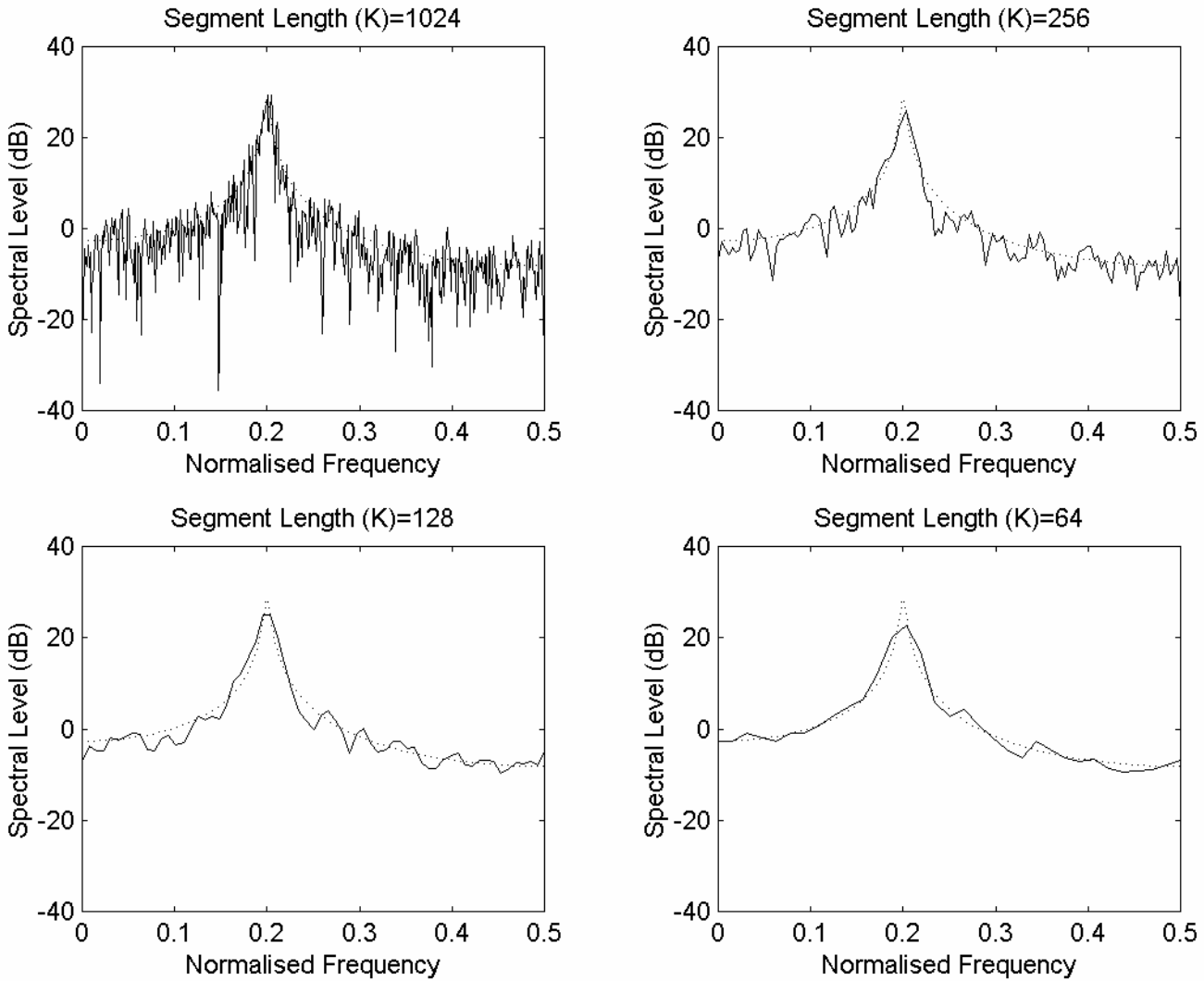
EXAMPLE OF BIAS/VARIANCE TRADE OFF FOR SEGMENT AVERAGING METHODS



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COMPARISON WITH PDF ESTIMATION

Spectral estimation has much in common with the problem of pdf estimation.

A spectrum and a pdf have very similar properties.

1) They are both non-negative.

2) The area under the functions are constrained:

pdf: $\int p(x)dx = 1$ and spectrum: $\int S(f)df = \sigma^2$

3) The are both densities:

pdf: $\int_a^b p(x)dx = P(a < X < b)$

spectrum: $\int_{f_0}^{f_1} S(f)df =$ Power in the band f_0 to f_1

The trade-offs involved in the estimation of these quantities are equivalent:

a) Wide bins/bandwidths result in smooth estimates, with large biases.

b) Narrow bins/bandwidths result in highly variable estimates.

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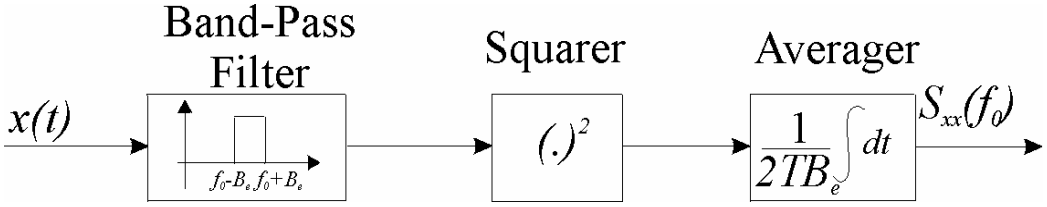
ESTIMATING THE SPECTRUM OF A TONAL SIGNAL

Let us consider the problem of estimating the spectrum of a signal that consists of the sum of pure sinusoids.

The theoretical spectrum contains only delta functions, so is discontinuous.

The level of the spectral estimate depends on the resolution of the analysis.

For simplicity consider the filter bank estimation algorithm:



If the value of B_e is increased no more energy is passed to the squarer (assuming no new components are introduced).

The value of $S_{xx}(f_0)$ will be reduced, because of the $1/B_e$ factor.

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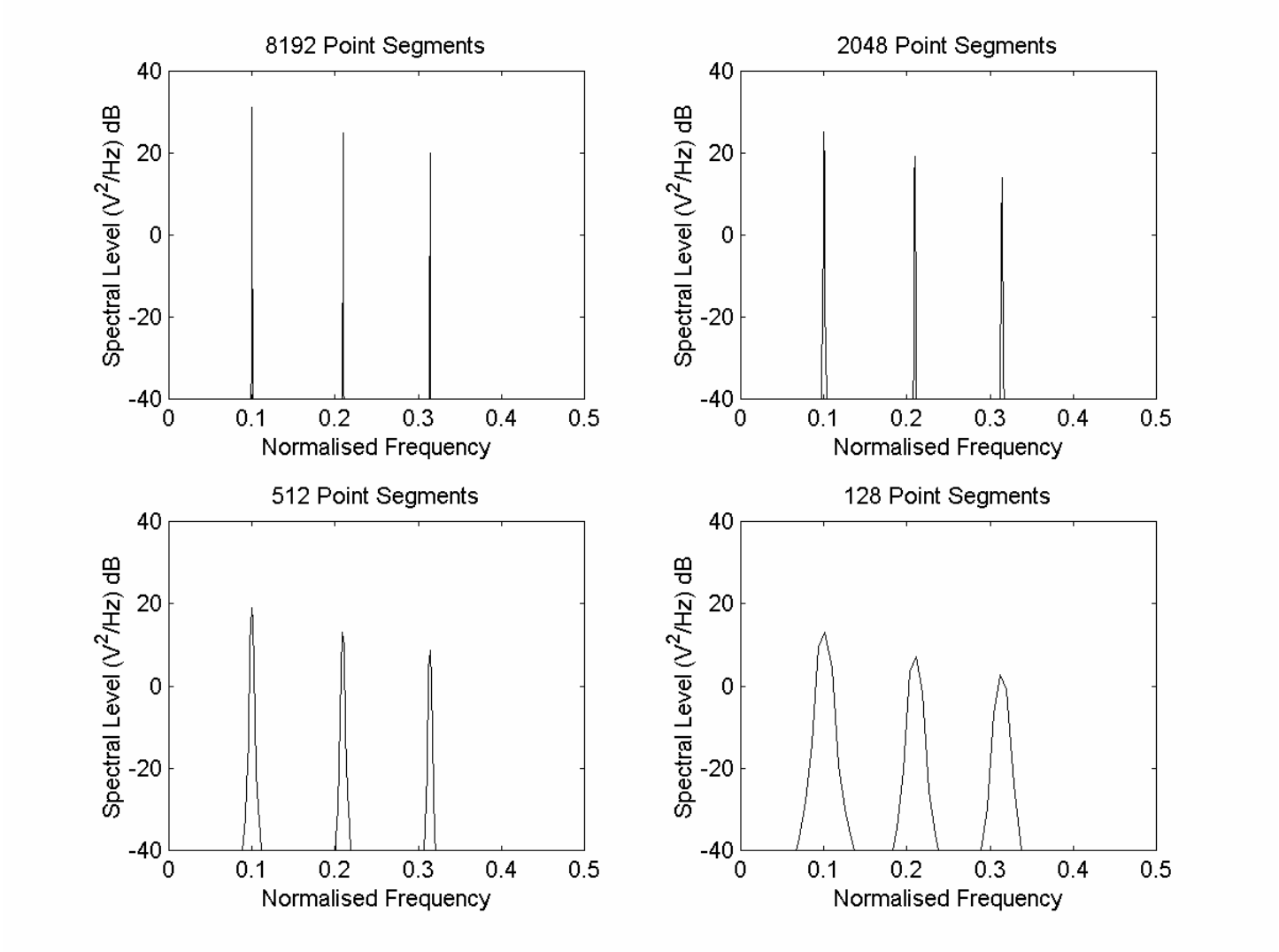
EXAMPLE OF SPECTRAL ANALYSIS OF PURE TONE SIGNALS



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AN APPROPRIATE NORMALISATION FOR TONAL SIGNALS

One can make the spectral estimate more appropriate to tonal signals by renormalisation.

Defining the spectral estimate as:

$$B_e \hat{S}(f)$$

Then for tonal signals this spectral estimate will be independent of the choice of analysis bandwidth, B_e.

Note this spectrum now has the units of v^2 rather than v^2/Hz.

Further by dividing the spectrum by appropriate normalisation (the exact form depends on the spectral estimator used) one can arrange for the level of the output spectrum to correspond to the squared amplitude of each sinusoidal term.

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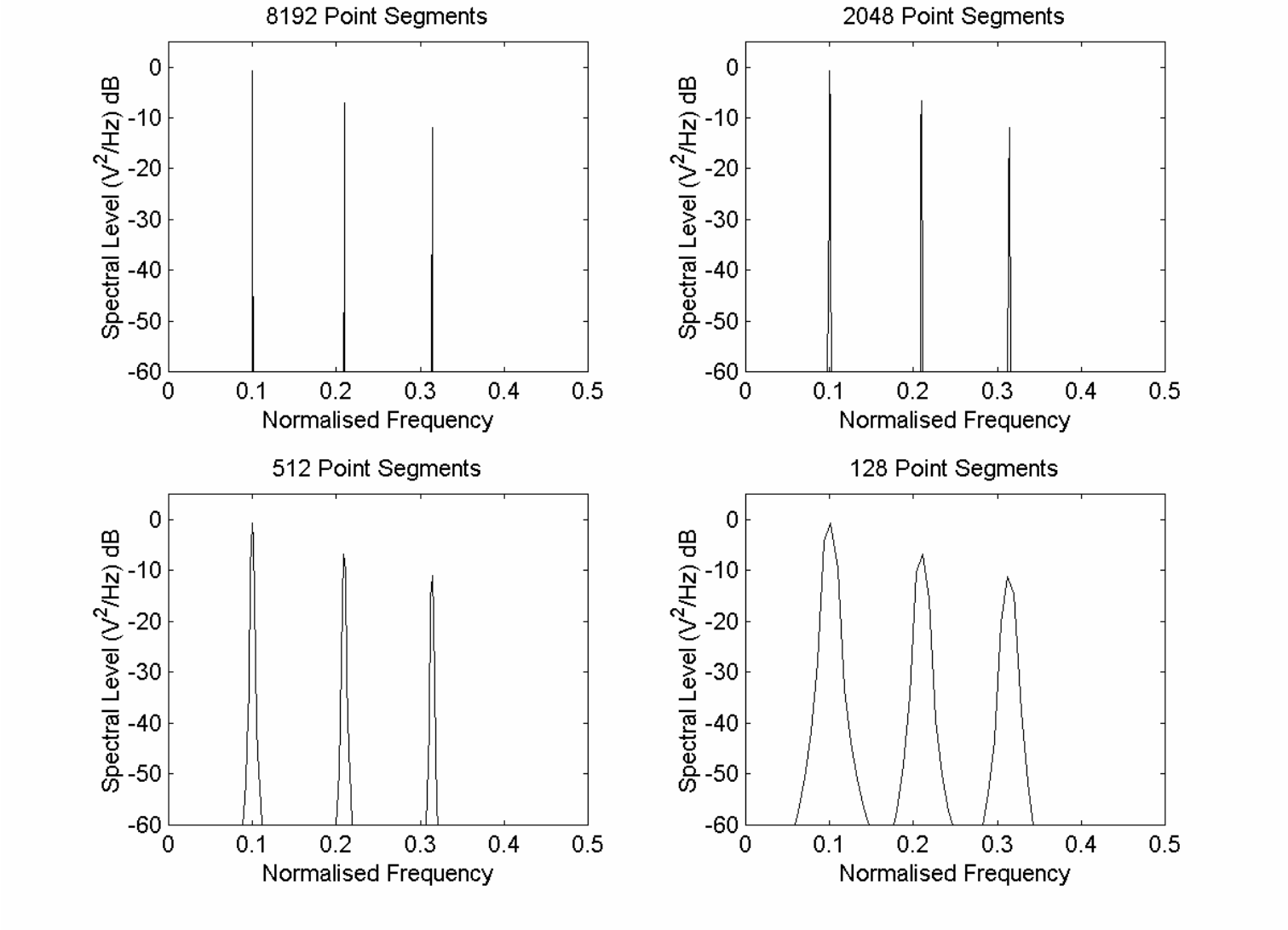
EXAMPLE OF SPECTRAL ESTIMATES CORRECTLY NORMALISED FOR TONAL AMPLITUDES



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MIXED PROCESSES

A mixed process in this context is signal whose spectrum is continuous almost everywhere, but does contain isolated Dirac delta functions, *i.e.* the signal is the sum of broadband noise and pure sinusoids.

Such signals are good models of signals encountered in many practical applications.

When processing such signals a basic problem arises as to which normalisation to use.

The height of the tones above the background noise depends on the resolution used.

If B_e is small then the tonal signals project above the background noise more than if B_e is large.

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EXAMPLE OF ESTIMATING THE SPECTRUM OF A MIXED PROCESS

